Fundamental Theorems in W^*-Algebras and the Kaplansky density theorem, III

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Abstract

Let $\mathscr{M}$ be a W^*-algebra and A a $\sigma(\mathscr{M}, V)$-dense *-subalgebra of $\mathscr{M}$. We shall give extraordinarily elementary proofs of the Kaplansky density theorem which are independent of the Kaplansky's that.

Let $\mathscr{M}$ be a W^*-algebra and V a uniformly dense linear subspace of $\mathscr{M}_*$ such that $\varphi^*, a\varphi$ and φa belong to V for every $\varphi \in V$ and $a \in \mathscr{M}$. Let A be a $\sigma(\mathscr{M}, V)$-dense *-subalgebra of $\mathscr{M}$. Kaplansky proved that the unit ball of the self-adjoint portion of A is σ-weakly dense in the unit ball of the self-adjoint portion of $\mathscr{M}$ and, in virtue of this fact and a discussion on matrices, proved that the unit ball of A is σ-weakly dense in the unit ball of $\mathscr{M}$. We shall give more easy proofs than the Kaplansky's proof of that the unit ball of the self-adjoint portion of A is σ-weakly dense in the self-adjoint portion of $\mathscr{M}$. There the continuity of the function is reduced to the continuity of the resolvents. Furthermore, from this fact, we can extraordinarily elementarily see the Kaplansky density theorem. The proofs are independent of the Kaplansky's that.

Let $\mathscr{M}_s$ and A_s denote the self-adjoint portions of $\mathscr{M}$ and A, respectively, and let $\mathscr{M}_+$ and A_+ denote the positive portions of $\mathscr{M}$ and A, respectively. Let $\mathscr{S}$ denote the unit ball of $\mathscr{M}$.

Lemma 1. *Let $\mathscr{M}$ and V be as above and A a $\sigma(\mathscr{M}, V)$-dense *-subalgebra of $\mathscr{M}$. Then $A_+ \cap \mathscr{S}$ is σ-weakly dense in $\mathscr{M}_+ \cap \mathscr{S}$ and so $A_s \cap \mathscr{S}$ is σ-strongly dense in $\mathscr{M}_s \cap \mathscr{S}$.*

Proof. A_s is $\tau(\mathscr{M}, V)$-dense in $\mathscr{M}_s$. Since $\|(x \pm i1)^{-1}\| \leq 1$ for every $x \in \mathscr{M}_s$, the functions $\mathscr{M}_s \ni x \mapsto (x \pm i1)^{-1} \in \mathscr{M}$ are continuous with respect to the $\tau(\mathscr{M}, V)$-topology and $\sigma(\mathscr{M}, V)$-topology. Notice that $(1 + x^2)^{-1} = 2^{-1}i((x + i1)^{-1} - (x - i1)^{-1})$ for every self-adjoint element x. Hence the function $\mathscr{M}_s \ni x \mapsto (1+x^2)^{-1} \in \mathscr{M}$ is continuous with respect to the $\tau(\mathscr{M}, V)$-topology and $\sigma(\mathscr{M}, V)$-topology. Since $(1 + x^2)^{-1}x^2 = 1 - (1 + x^2)^{-1}$, the function $\mathscr{M}_s \ni x \mapsto (1+x^2)^{-1}x^2 \in \mathscr{M}$ are continuous with respect to the $\tau(\mathscr{M}, V)$-topology and σ-weak topology. The range of this function coincides with the set $\{x \in \mathscr{M}_+ \mid \|x\| < 1\}$. Therefore we have $\{x \in \mathscr{M}_+ \mid \|x\| < 1\} \subset \overline{A_+ \cap \mathscr{S}}$ and so $\mathscr{M}_+ \cap \mathscr{S} \subset \overline{A_+ \cap \mathscr{S}}$. If $0 \leq a \leq 1$

and $0 \leq b \leq 1$, then we have $\|a-b\| \leq 1$. For any self-adjoint element x in $\mathscr{S}$, the positive part and negative part of x belong to $\overline{A_+ \cap \mathscr{S}}$. Therefore we have $x \in \overline{A_s \cap \mathscr{S}}$. □

Let V be as above; then we call the locally convex topology on $\mathscr{M}$ defined by seminorms $p_\varphi(x) = \varphi(x^*x)^{1/2}$ with $0 \leq \varphi \in V$ the V-strong topology. Define the seminorm p^*_φ by $p^*_\varphi(x) = \varphi(xx^*)^{1/2}$ with $0 \leq \varphi \in V$ and we call the locally convex topology defined by all p_φ and p^*_φ the V-strong* topology. Since V is linearly spanned by positive elements of V in virtue of Jordan decomposition, the V-strong topology is Hausdorff and finer than the $\sigma(\mathscr{M}, V)$-topology.

The function $\mathscr{M}_s \ni x \mapsto \frac{1}{2}i((x+i1)^{-1} - (x-i1)^{-1}) = (1+x^2)^{-1} \in \mathscr{M}_s$ is V-strongly continuous. Notice that $2x(1+x^2)^{-1} = (x+i1)^{-1} + (x-i1)^{-1}$ for every self-adjoint element x. Hence it is trivial that the function $\mathscr{M}_s \ni x \mapsto 2x(1+x^2)^{-1} \in \mathscr{M}_s$ is V-strongly continuous and continuous with respect to the $\tau(\mathscr{M}, V)$-topology and σ-weak topology.

A continuous complex-valued function f on a closed subset I of $\mathbf{R}$ is said to be V-strongly (resp., σ-strongly) continuous if the function $x \mapsto f(x) \in \mathscr{M}$ defined for a self-adjoint element x with $\mathrm{Sp}(x) \subset I$ is V-strongly (resp., σ-strongly) continuous. If I is bounded, then f may be approximated uniformly by polynomials p_n. It follows that, for any self-adjoint element x with $\mathrm{Sp}(x) \subset I$,

$$\|p_n(x) - f(x)\| \leq \sup_{t \in I} |p_n(t) - f(t)| \to 0 \quad \text{as} \quad n \to \infty,$$

in virtue of Gelfand representation. Since the function $x \mapsto p_n(x) \in \mathscr{M}$ is V-strongly and σ-strongly continuous, the function $x \mapsto f(x) \in \mathscr{M}$ is V-strongly and σ-strongly continuous.

Theorem 2 (Kaplansky). *Let $\mathscr{M}$ and V be as above and A a *-subalgebra of $\mathscr{M}$ which is $\sigma(\mathscr{M}, V)$-dense in $\mathscr{M}$. Then the unit ball of A is $\tau(\mathscr{M}, \mathscr{M}_*)$-dense in the unit ball of $\mathscr{M}$.*

Proof. We may assume that A is uniformly closed. By Lemma 1, any element x in the unit ball of $\mathscr{M}$ belongs to the σ-strong* closure of $A \cap 2\mathscr{S}$. Let f_ε be a continuous function on the interval $[0,4]$ such that $f_\varepsilon(t) = (1+\varepsilon)^{-1/2}$ on $[0,1]$, $\mathrm{Supp}(f_\varepsilon) \subset [0, 1+\varepsilon]$ and $0 \leq f_\varepsilon \leq (1+\varepsilon)^{-1/2}$. Since f_ε is σ-strongly continuous, the function $2\mathscr{S} \ni y \to f_\varepsilon(yy^*)y$ is continuous with respect to the σ-strong* topology and the σ-strong topology. Hence $f_\varepsilon(xx^*)x$ belongs to the σ-strong closure of the image of $A \cap 2\mathscr{S}$ under this function. Since $\|f_\varepsilon(yy^*)yy^*f_\varepsilon(yy^*)\| \leq 1$, we have $\|f_\varepsilon(yy^*)y\| \leq 1$. Since $f_\varepsilon(yy^*)y \in A$ for every $y \in A$, $f_\varepsilon(xx^*)x$ belongs to the σ-strong closure of the unit ball of A. On the other hand, we have $f_\varepsilon(xx^*) = (1+\varepsilon)^{-1/2}1$. Therefore x belongs to the σ-strong closure of the unit ball of A. □

Under the assumption that A is V-strongly* dense in $\mathscr{M}$ or in virtue of Lemma 13 in [2], by using a similar function g_ε as f_ε, we can see Lemma 1 and Theorem 2. Let g_ε be a continuous function on the interval $[0,1]$ such that $g_\varepsilon(t) = (1+\varepsilon)^{-1/2}$ on $[1/2, 1]$, $\mathrm{Supp}(g_\varepsilon) \subset [(2+\varepsilon)^{-1}, 1]$ and $0 \leq g_\varepsilon \leq (1+\varepsilon)^{-1/2}$. Let x be a self-adjoint element of the unit ball of $\mathscr{M}$. Since the functions g_ε and $\mathscr{M}_s \ni y \mapsto (1+y^2)^{-1} \in \mathscr{M}_s$ are V-strongly (σ-strongly) continuous, the function $\mathscr{M}_s \ni y \mapsto g_\varepsilon((1+y^2)^{-1})y$ is V-strongly (σ-strongly) continuous. Hence $g_\varepsilon((1+x^2)^{-1})x$ belongs to the V-strong (σ-strong) closure of the image of A_s under this function. Since $\|g_\varepsilon((1+yy^*)^{-1})yy^*g_\varepsilon((1+yy^*)^{-1})\| \leq 1$, we have $\|g_\varepsilon((1+yy^*)^{-1})y\| \leq 1$. Therefore $g_\varepsilon((1+x^2)^{-1})x$ belongs to the V-strong (σ-strong) closure of the unit ball of A_s. On the other hand, we have $g_\varepsilon((1+x^2)^{-1}) = (1+\varepsilon)^{-1/2}1$. Therefore x belongs to the V-strong (σ-strong) closure of the unit ball of A_s.

Next, let x be an element of the unit ball of $\mathscr{M}$; then x belongs to the V-strong (σ-strong) closure of $A \cap 2\mathscr{S}$. Since the function $\mathscr{M}_+ \ni y \mapsto (1+y)^{-1} \in \mathscr{M}_+$ is V-strongly (σ-strongly) continuous, the function $2\mathscr{S} \ni y \mapsto g_\varepsilon((1+yy^*)^{-1})y \in \mathscr{S}$ is V-strongly* (σ-strongly*) continuous. Hence $g_\varepsilon((1+xx^*)^{-1})x$ belongs to the V-strong (σ-strong) closure of the unit ball of A. Therefore x belongs to the V-strong (σ-strong) closure of the unit ball of A and so belongs to the $\sigma(\mathscr{M}, V)$-closure of the unit ball of A. Hence x belongs to the σ-weak closure of the unit ball of A and so belongs to the $\tau(\mathscr{M}, \mathscr{M}_*)$-closure of the unit ball of A.

Using the V-strong* continuity of the function $\mathscr{M} \ni y \mapsto (1+yy^*)^{-1} \in \mathscr{M}_+$, we can directly see that the function $\mathscr{M} \ni y \mapsto g_\varepsilon((1+yy^*)^{-1})y \in \mathscr{M}$ is continuous with respect to the V-strong* (σ-strong*) topology and V-strong (σ-strong) topology.

In this proof of the Kaplansky density theorem, we does not discuss on the range of a V-strongly continuous function and use only the V-strong continuity of functions.

Proposition 3. *Let $\mathscr{M}$ be a W^*-algebra. A continuous complex-valued function f on* $\mathbf{R}$ *such that $|f(t)| \leq \alpha|t| + \beta$ for some non negative numbers α and β is V-strongly and σ-strongly continuous.*

Proof. The functions $\mathscr{M}_s \ni x \mapsto (i1 \pm x)^{-1} \in \mathscr{M}$ are V-strongly and σ-strongly continuous. Put $g(t) = (i+t)^{-1}$ for each $t \in \mathbf{R}$; then g is V-strongly and σ-strongly continuous. Extending g to the function $\bar{g}$ on the one-point compactification $\mathbf{R} \cup \{\omega\}$ by $\bar{g}(\omega) = 0$, $\bar{g}$ is injective and continuous. Therefore any continuous function on $\mathbf{R} \cup \{\omega\}$ can be uniformly approximated by polynomials of $\bar{g}$, the complex conjugate $\bar{g}^*$ of $\bar{g}$ and the constant function 1, in virtue of the Stone-Weierstrass theorem. Hence any continuous function in $C_0(\mathbf{R})$ can be uniformly approximated by polynomials of g and g^*, and so is V-strongly and σ-strongly continuous.

If f is a bounded continuous function on $\mathbf{R}$, then the function $t \mapsto f(t)(i+t)^{-1}$ belongs to $C_0(\mathbf{R})$ and so is V-strongly and σ-strongly continuous. Hence the function $\mathscr{M}_s \ni x \mapsto (f(x)(i1+x)^{-1})(i1+x) = f(x) \in \mathscr{M}$ is V-strongly and σ-strongly continuous. If f is a continuous complex-valued function on $\mathbf{R}$ such that $|f(t)| \leq \alpha|t| + \beta$ for some non negative numbers α and β, then the function $t \mapsto f(t)(i+t)^{-1}$ is bounded and so V-strongly and

σ-strongly continuous. Hence the function $t \mapsto (f(t)(i+t)^{-1})(i+t) = f(t)$ is V-strongly and σ-strongly continuous. $\square$

Under the assumption that A is V-strongly* dense in $\mathscr{M}$ or in virtue of Lemma 13 in [2], A_s is V-strongly dense in $\mathscr{M}_s$. By Proposition 3, the continuous function $\mathbf{R} \ni t \mapsto (t \wedge 1) \vee (-1)$ is V-strongly continuous. Therefore, for any element x in the unit ball of $\mathscr{M}_s$, we have $x = (x \wedge 1) \vee (-1) = \lim_{y \to x, y \in A_s} (y \wedge 1) \vee (-1)$, so that x belongs to the V-strong closure of the unit ball of A_s. By (I), the unit ball of A is V-strongly dense in the unit ball of $\mathscr{M}$ and so $\sigma(\mathscr{M}, V)$-dense in the unit ball of $\mathscr{M}$. Therefore the unit ball of A is σ-weakly dense in the unit ball of $\mathscr{M}$.

References

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